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AN APPLICATION OF NUMBER THEORY TO ERGODIC THEORY AND THE CONSTRUCTION OF UNIQUELY ERGODIC MODELS

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To the memory of Shlomo Horowitz

ABSTRACT

Using a combinatorial result of N. Hindman one can extend Jewett's method for proving that a weakly mixing measure preserving transformation has a uniquely ergodic model to the general ergodic case. We sketch a proof of this reviewing the main steps in Jewett's argument.

§1. Introduction

The Jewett–Krieger theorem on the existence of uniquely ergodic models may be stated as follows:

THEOREM (Jewett–Krieger). *Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving system where $(X, \mathcal{A}, \mu)$ is a Lebesgue space and $T: X \to X$ an invertible ergodic measure-preserving transformation. Then this ergodic system is isomorphic to a uniquely ergodic dynamical system (Y, U), where Y is the Cantor set and U is a homeomorphism of Y which leaves invariant precisely one Borel probability measure.*

The pioneering work was done by Robert I. Jewett in [7]. He proved the theorem under *the additional assumption* that T is *weakly mixing*. Jewett's paper was considered an important breakthrough when it first came out. It is also a model of lucidity and elegance. The ideas are simple and natural, the proofs straightforward; as a matter of fact the main tools used for the most part of the

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paper are the Individual Ergodic Theorem and the notion of Kakutani–Rokhlin Skyscraper ("building" in Jewett's terminology). In analyzing Jewett's proof it turns out that the *weakly mixing* character of T is *used only once* in the paper, namely in the proof of the Lemma on p. 720 (see lines 2, 3 from below); everywhere else in the paper ergodicity suffices. To facilitate matters we shall refer to this Lemma as the *Key Lemma* in Jewett's paper.

In the introduction to his paper Jewett made the following statement: "It seems likely that if the condition of being weakly mixing were replaced by that of being ergodic, the theorem would still be valid." Jewett's conjecture was proved by Krieger in [8]. This was followed by the papers of Hansel and Raoult [5] and Denker [1], giving different proofs of the theorem in the general ergodic case. (See also [2]; a detailed account of the history of the Jewett–Krieger theorem can be found on pp. 300–301.) It is worth noting that each one of these papers is substantially longer and more elaborate than Jewett's original paper. These papers also use heavier machinery and sometimes tools that are extraneous to the problem, such as entropy, the Shannon–McMillan Theorem, the existence of a finite generator, intrinsic Markov chains (= subshifts of finite type), etc.

The purpose of the present paper is twofold. In §2 we give a new result concerning abstract measure-preserving systems (Theorem 2). This is of independent interest. In the second part of the paper we show how with this additional piece of information the Key Lemma in Jewett's paper — and hence Jewett's whole proof — carries over to the general ergodic case. This is done in §4. To facilitate the reader's task we include in §3 a review of the notion of Kakutani–Rokhlin Skyscraper as well as the preliminary lemmas in Jewett's paper.

§2. A combinatorial result concerning measure-preserving systems

Throughout this section, $(X, \mathcal{A}, \mu, T)$ is a *measure-preserving system*, i.e., $(X, \mathcal{A}, \mu)$ is a probability space and $T: X \to X$ a measurable, measure-preserving transformation.

By a *finite partition of* X, we shall mean a finite collection $\mathcal{C} = \{C_1, \cdots, C_s\}$ of disjoint measurable sets, whose union covers X up to a set of μ-measure zero.

The *gauge of the partition* $\mathcal{C}$ is defined as

$$\delta(\mathcal{C}) = \inf\{\mu(C_i) \mid C_i \in \mathcal{C} \text{ and } \mu(C_i) > 0\}.$$

If $\mathcal{D}$ is another *finite partition of* X, then $\mathcal{C} \vee \mathcal{D}$ denotes their common refinement.

In particular, we may consider $\mathscr{C} \vee T^{-n}(\mathscr{C})$ and look at the asymptotic behavior of $\delta(\mathscr{C} \vee T^{-n}(\mathscr{C}))$ as $n \to \infty$. This is the object of the main result of this section, Theorem 2.

In what follows we denote the set of natural numbers $\{1, 2, 3, \cdots\}$ by $\mathbf{N}$ and we denote by $\mathscr{F}(\mathbf{N})$ the collection of all finite subsets of $\mathbf{N}$.

Let $p_1, p_2, \cdots, p_n, \cdots$ be a sequence of elements in $\mathbf{N}$. We introduce the following notation: For each $\alpha \in \mathscr{F}(\mathbf{N})$, $\alpha \neq \emptyset$, say $\alpha = \{i_1, i_2, \cdots, i_k\}$, we write

$$p_\alpha = p_{i_1} + p_{i_2} + \cdots + p_{i_k}.$$

We now recall the notion of an *IP-set* (see [3] for a detailed study):

Definition 2.1. A subset $S \subset \mathbf{N}$ is called an *IP-set* if there exists a sequence $p_1, p_2, p_3, \cdots, p_n, \cdots$ of distinct elements of S such that S coincides with "the set of all finite sums"

$$S = \{p_\alpha \mid \alpha \in \mathscr{F}(\mathbf{N}),\ \alpha \neq \emptyset\}.$$

S is said to be the IP-set generated by the sequence $p_1, p_2, p_3, \cdots, p_n, \cdots$.

Remark. Let $p_1, p_2, p_3, \cdots, p_n, \cdots$ be a sequence of distinct elements of $\mathbf{N}$. If we extract a subsequence, then it is clear that the IP-set generated by this new sequence is contained in the IP-set generated by $p_1, p_2, \cdots, p_n, \cdots$.

Lemma 2.1. *Let $n_1 < n_2 < n_3 < \cdots$ be a strictly increasing sequence of natural numbers. Let $A \in \mathscr{A}$ with $\mu(A) > 0$. Then for any $\varepsilon > 0$ there exist infinitely many pairs (i, j) with $i < j$ for which*

$$\mu(A \cap T^{-(n_j - n_i)}(A)) > \mu(A)^2 - \varepsilon.$$

Proof. We reason by contradiction. Suppose that for all (i, j), $i_0 \leqq i < j$, $\mu(A \cap T^{-(n_j - n_i)}(A)) \leqq \mu(A)^2 - \varepsilon$, whence

$$\mu(T^{-n_i}(A) \cap T^{-n_j}(A)) \leqq \mu(A)^2 - \varepsilon.$$

Let $f = 1_A - \mu(A)$ and compute

$$0 \leqq \int [f(T^{n_{i_0}}x) + f(T^{n_{i_0+1}}x) + \cdots + f(T^{n_{i_0+p-1}}x)]^2 d\mu(x)$$

$$= \int [1_{T^{-n_{i_0}}(A)} + \cdots + 1_{T^{-n_{i_0+p-1}}(A)} - p\mu(A)]^2 d\mu$$

$$\leqq p(p-1)(\mu(A)^2-\varepsilon)+p\mu(A)+p^2\mu(A)^2-2p^2\mu(A)^2.$$

This last expression clearly tends to $-\infty$ as $p \to +\infty$. This proves the lemma.

LEMMA 2.2. *Let $S \subset \mathbf{N}$ be an* IP-*set. Let $A \in \mathcal{A}$ with $\mu(A)>0$. Then for any $\varepsilon>0$ there are infinitely many elements $s \in S$ for which*

$$\mu(A \cap T^{-s}(A)) > \mu(A)^2-\varepsilon.$$

PROOF. We may assume without loss of generality that S is generated by a sequence satisfying the growth condition $p_{k+1} > p_1+p_2+\cdots+p_k$, for each $k \in \mathbf{N}$ (otherwise extract a subsequence; see Remark above). Now set $n_k = p_1+\cdots+p_k$ for each $k \in \mathbf{N}$. Note that for each pair (i,j) with $i<j$ the difference $n_j-n_i = p_{i+1}+p_{i+2}+\cdots+p_j$ belongs to S. Also if (i,j) and if (i',j') are distinct pairs with $j<j'$, then $n_{j'}-n_{i'} > n_j - n_i$. The proof is concluded by applying Lemma 2.1.

THEOREM 1. *Let $A, B \in \mathcal{A}$, let $S \subset \mathbf{N}$ be an* IP-*set and assume that $\mu(A \cap T^{-s}(B)) = \varepsilon(s) > 0$ for each $s \in S$. Then we cannot have $\varepsilon(s) \to 0$ as $s \to \infty$.*

PROOF. Let $q_1, q_2, q_3, \cdots, q_n, \cdots$ generate S. We have

$$\mu(A \cap T^{-q_1}(B)) = \varepsilon(q_1) > 0.$$

Now apply Lemma 2.2 to the set $A' = A \cap T^{-q_1}(B)$ and to the IP-set R generated by $q_2, q_3, \cdots$. For infinitely many elements $r \in R$ we have

$$\mu(A' \cap T^{-r}(A')) > \varepsilon(q_1)^2/2.$$

Note that if $r \in R$ then $q_1 + r \in S$ and that

$$A \cap T^{-(r+q_1)}(B) \supset A' \cap T^{-r}(A').$$

We deduce $\mu(A \cap T^{-(r+q_1)}(B)) > \varepsilon(q_1)^2/2$ for infinitely many $r \in R$, that is

$$\varepsilon(r+q_1) > \varepsilon(q_1)^2/2.$$

This proves the theorem.

At this stage we need to recall a theorem from combinatorial number theory due to Hindman [6]:

THEOREM (Hindman). *For any finite partition of* $\mathbf{N}$, $\mathbf{N} = H_1 \cup H_2 \cup \cdots \cup H_n$, *one of the sets H_j contains an* IP-*set.*

A proof of this result using ultrafilters may be found in [4]. Still another proof based on topological dynamics appears in [3].

THEOREM 2. *Let $\mathscr{C} = \{C_1, \cdots, C_p\}$ be a finite partition of X. Then*

$$\limsup_n \delta(\mathscr{C} \vee T^{-n}(\mathscr{C})) > 0.$$

PROOF. Let

$$Q = \{(i,j) \mid \text{for some } n \in \mathbf{N},\ \mu(C_i \cap T^{-n}(C_j)) = \delta(\mathscr{C} \vee T^{-n}(\mathscr{C}))\}$$

and for each $(i,j) \in Q$ let

$$H(i,j) = \{n \in \mathbf{N} \mid \delta(\mathscr{C} \vee T^{-n}(\mathscr{C})) = \mu(C_i \cap T^{-n}(C_j))\}.$$

This yields a finite partition of $\mathbf{N}$,

$$\mathbf{N} = \bigcup_{(i,j)\in Q} H(i,j).$$

By Hindman's theorem, there is $(i,j) \in Q$ such that $H(i,j)$ contains an IP-set. Applying Theorem 1 to C_i, C_j and this IP-set completes the proof.

§3. A brief review of Kakutani–Rokhlin skyscrapers

In the remainder of this paper we assume that $(X, \mathscr{A}, \mu, T)$ is a measure-preserving system with $(X, \mathscr{A}, \mu)$ a *Lebesgue space* and $T : X \to X$ an *invertible ergodic measure-preserving transformation.*

Since the set of periodic points (points $x \in X$ whose orbit under T is finite) is easily seen to be measurable and T-invariant, *we may and shall assume that the set of periodic points is void.*

We now recall the notion of "Kakutani–Rokhlin skyscraper" ("building" in Jewett's terminology). We begin with the notion of "block":

DEFINITION 3.1. A block is a set of the form

$$B = \{x, Tx, \cdots, T^{n-1}x\}$$

for some $x \in X$ and $n \in \mathbf{N}$. The element x is called the initial element of the block, the element $T^{n-1}x$ is called the last element of the block and the integer n is called the height of the block B.

DEFINITION 3.2. A Kakutani–Rokhlin skyscraper, or simply a skyscraper, is a collection $\mathbf{B}$ of disjoint blocks such that

(i) For each $n \in \mathbf{N}$,

$$F_n = \{x \in X \mid \{x, Tx, \cdots, T^{n-1}x\} \in \mathbf{B}\},$$

i.e. the base of blocks of height n, is a set in $\mathcal{A}$.

(ii) $\Sigma_{n \in \mathbf{N}} n\mu(F_n) = 1$, i.e. the skyscraper covers X up to a set of μ-measure zero.

Before stating the next lemma we need some more notation.

For each finite set $A \subset X$, $f \in L^1$, $n \in \mathbf{N}$, we write

$$|A| = \text{cardinality of } A,$$

$$\mathrm{av}_A f = \frac{1}{|A|} \sum_{x \in A} f(x),$$

$$T_n f = \frac{1}{n} \sum_{k=0}^{n-1} f_\circ T^k.$$

LEMMA 3.1. *Let $f \in L^1$, $\varepsilon > 0$ and $p \in \mathbf{N}$. There is then a skyscraper $\mathbf{B}$ and an integer $q > p$ such that*

(a) *For every $B \in \mathbf{B}$, either $|B| = p$ or $p < |B| \leqq q$.*

(b) *For every $B \in \mathbf{B}$ with $|B| > p$,*

$$\left| \int_X f d\mu - \mathrm{av}_B f \right| < \varepsilon.$$

(c) $\mu\left(\bigcup_{\substack{B \in \mathbf{B} \\ |B| = p}} B \right) < \varepsilon.$

For a proof see Jewett's paper, p. 719.

§4. Proof of the Key Lemma in Jewett's paper in the general ergodic case

The Key Lemma in Jewett's paper is the following:

KEY LEMMA (Jewett). *Let $f : X \to R$ be a simple function, let $\varepsilon > 0$ and $m \in \mathbf{N}$. Then there exists a simple function $g : X \to R$ such that:*

(i) $\mu(\{x \mid f(x) \neq g(x)\}) < \varepsilon$;

(ii) $\|\int_X f d\mu - T_n g\|_\infty < \varepsilon$ *for all sufficiently large* $n \in \mathbf{N}$;

(iii) *The set* $\{(g(x), g(Tx), \cdots, g(T^{m-1}x)) \mid x \in X\}$ *is contained in the set* $\{(f(x), f(Tx), \cdots, f(T^{m-1}x)) \mid x \in X\}$.

PROOF. The vector-valued function $F: X \to R^m$,

$$x \to F(x) = (f(x), f(Tx), \cdots, f(T^{m-1}x)),$$

assumes only a finite number of values with strictly positive probability, say a_i, $1 \leqq i \leqq r$. Let

$$E_i = \{x \in X \mid F(x) = a_i\}, \qquad 1 \leqq i \leqq r.$$

Then $\mathscr{C} = \{E_1, \cdots, E_r\}$ is a finite partition of X in the sense of §2. By Theorem 2 (see §2),

$$\limsup_p \delta(\mathscr{C} \vee T^{-p}(\mathscr{C})) > 0.$$

Hence there is $0 < \alpha < 1$ such that for *infinitely many* $p \in \mathbf{N}$ the following holds:

$$\left\{\begin{array}{c} 1 \leqq i, j \leqq r \\ \text{and} \\ \mu(E_i \cap T^{-p}(E_j)) > 0 \end{array}\right\} \Rightarrow \mu(E_i \cap T^{-p}(E_j)) \geqq \alpha. \tag{1}$$

For each $n \in \mathbf{N}$ let

$$A_n = \left\{ x \in X \,\middle|\, \left| \int_X f d\mu - T_k f(x) \right| < \frac{\varepsilon}{2} \qquad \text{for all } k \geqq n \right\}.$$

By the Individual Ergodic Theorem, $\mu(A_n) \nearrow 1$. *Choose* $n > m$ large enough that

$$\mu(A_n) > 1 - \alpha \tag{2}$$

and *then choose* $p > n$ such that (1) holds.

Let now

$$K = \{(i, j) \mid 1 \leqq i, j \leqq r, \quad \text{and} \quad \mu(E_i \cap T^{-p}(E_j)) > 0\}$$

and denote

$$E_{ij} = E_i \cap T^{-p}(E_j), \qquad \text{for } (i, j) \in K.$$

Then $\{E_{ij} \mid (i, j) \in K\}$ is a *finite partition* of X, $\mu(E_{ij}) \geqq \alpha$ for all $(i, j) \in K$ and hence by (2):

(3) $$\mu(A_n \cap E_{ij}) > 0 \qquad \textit{for all } (i,j) \in K.$$

By Lemma 3.1 applied to f, $\varepsilon/2$, p, there is a skyscraper $\mathbf{B}$ and an integer $q > p$ such that:

(α) For every $B \in \mathbf{B}$, either $|B| = p$ or $p < |B| \leqq q$.

(β) For every $B \in \mathbf{B}$ with $|B| > p$,

$$\left| \int_X f d\mu - \mathrm{av}_B f \right| < \frac{\varepsilon}{2}.$$

$$(\gamma)\ \mu\Bigg(\bigcup_{\substack{B \in \mathbf{B} \\ |B| = p}} B \Bigg) < \frac{\varepsilon}{2}.$$

We may assume in addition (see Lemma 3.1) that

(δ) $T\Big(\bigcup_{B \in \mathbf{B}} B \Big) = \bigcup_{B \in \mathbf{B}} B \subset \bigcup_{(i,j) \in K} E_{ij}$.

For *each* $(i,j) \in K$ *pick* an element $y_{ij} \in E_{ij} \cap A_n$. We may now define g as follows: On the complement of $\bigcup_{B \in \mathbf{B}} B$ we let $g = f$. On the skyscraper $\mathbf{B}$ we define g as follows:

(4) $$g = f \qquad \text{on each } B \in \mathbf{B} \text{ with } |B| > p.$$

If $B \in \mathbf{B}$ and $|B| = p$, let x be the initial element of the block B; by (δ), $x \in E_{ij}$ for a *unique* $(i,j) \in K$. Define g on B by

(5) $$(g(x), g(Tx), \cdots, g(T^{p-1}x)) = (f(y_{ij}), f(Ty_{ij}), \cdots, f(T^{p-1}y_{ij})).$$

Because $y_{ij} \in A_n$ and $p > n$, we have

$$\left| \int_X f d\mu - \mathrm{av}_B g \right| < \frac{\varepsilon}{2}.$$

Hence (see (β) and (4) above) *for every block* $B \in \mathbf{B}$ we have

$$\left| \int_X f d\mu - \mathrm{av}_B g \right| < \frac{\varepsilon}{2}.$$

Since the function g is bounded and the blocks of $\mathbf{B}$ are bounded in cardinality, we deduce that (ii) holds.

Since

$$\{x \mid g(x) \neq f(x)\} \subset \bigcup_{\substack{B \in \mathbf{B} \\ |B| = p}} B$$

we deduce (see (γ) above) that (i) holds.

It remains to check (iii). Since $g = f$ on

$$C\Big(\bigcup_{B\in\mathbf{B}} B\Big) = T\Big(C\Big(\bigcup_{B\in\mathbf{B}} B\Big)\Big),$$

we need only be concerned with what happens on blocks $B \in \mathbf{B}$.

Note first that if $u, v \in E_j$ then

$$(6) \qquad (f(u), f(Tu), \cdots, f(T^{m-1}u)) = a_j = (f(v), f(Tv), \cdots, f(T^{m-1}v)).$$

Let now $B \in \mathbf{B}$ and let x be the initial element of the block B. There are two possibilities: (I) $|B| = p$; (II) $|B| > p$.

(I) $|B| = p$. Assume $x \in E_{ij} = E_i \cap T^{-p}(E_j)$; then (see (5))

$$(7) \qquad (g(x), \cdots, g(T^{p-1}x)) = (f(y_{ij}), \cdots, f(T^{p-1}y_{ij})).$$

Also T^p_x and $T^p_{y_{ij}}$ both belong to E_j. There are two cases:

Case I. 1. T^p_x is the initial element of a block of height $> p$. Then since g and f coincide on this block we have (use also (6)):

$$\begin{aligned}(g(T^px), \cdots, g(T^{p+m-1}x)) = (f(T^px), \cdots, f(T^{p+m-1}x)) &= a_j \\ &= (f(T^p_{y_{ij}}), \cdots, f(T^{p+m-1}y_{ij})).\end{aligned}$$

Case I. 2. T^px is the initial element of a block of height p. Assume $T^px \in E_{jk} = E_j \cap T^{-p}(E_k)$; then since $y_{jk} \in E_{jk}$ we have

$$\begin{aligned}(g(T^px), \cdots, g(T^{p+m-1}x)) = (f(y_{jk}), \cdots, f(T^{m-1}y_{jk})) &= a_j \\ &= (f(T^py_{ij}), \cdots, f(T^{p+m-1}y_{ij})).\end{aligned}$$

In either Case I.1 or I.2 we have (use (7))

$$(8) \qquad (g(x), \cdots, g(T^{p+m-1}x)) = (f(y_{ij}), \cdots, f(T^{p+m-1}y_{ij})).$$

Similar reasoning shows that in Case II, if $|B| = s > p$ then

$$(9) \qquad (g(x), \cdots, g(T^{s+m-1}x)) = (f(x), \cdots, f(T^{s+m-1}x)).$$

From (8), (9) we derive (iii). This completes the proof of the Key Lemma.

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